HOMEWORK 3

PIETRO POGGI-CORRADINI

1. Assume $\operatorname{Re} z \leq 0$ and $\operatorname{Re} w \leq 0$. Show that

$$|e^z - e^w| \leq |z - w|.$$

Hint: use the complex-valued function $F(t) = e^{z+t(w-z)}$.

2. Find the series expansion of

$$\frac{z + 2i}{(z-2)(z^2+1)}$$

about the point 1.

Hint: use the partial fractions decomposition first and then the geometric series.

3. Suppose f is analytic in a connected open set U. If $|f(z)|$ is constant on U, prove that f is constant on U. Likewise, prove that f is constant if $\operatorname{Re} f$ is constant.

4. Suppose $|z| < 1$ and $|w| < 1$. Show that

$$\left| \frac{z - w}{1 - \bar{w} z} \right| < 1.$$

Hint: square both sides.

5. Show that the Joukowski map is just $f(z) = z^2$ preceeded by a "rotation of the unit sphere". Hint: do a ninety degree rotation around the x_2 axis.

6. Prove that if f is analytic on a region Ω then

$$\sup_{z \in \Omega} |f(z)| = \limsup_{z \to \partial\Omega} |f(z)|.$$

Hint: We say that a sequence tends to $\partial\Omega$ if it is eventually outside each compact subset of Ω. The lim sup is then the largest subsequential limit of the values of $|f|$ over all sequences tending to $\partial\Omega$. If Ω is unbounded, we view our sets as lying on the Riemann sphere, so that the boundary includes the North Pole (the point "at ∞"). Equivalently we can measure the distance to the boundary using the chordal metric.

7. Prove that if f is a one-to-one (two-to-two!) analytic map of an open set Ω onto $f(\Omega)$ and if $z_n \in \Omega \to \partial\Omega$, then $f(z_n) \to \partial f(\Omega)$, in the sense that $f(z_n)$ eventually lies outside each compact subset of $f(\Omega)$. Another way to state the problem is to view the sets as lying on the Riemann sphere, so that the boundary can include the North Pole (the point at "∞").

8*. (a) Define $n^{-z} = e^{-z \ln(n)}$. Prove that

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

converges uniformly and absolutely in $\{z : \operatorname{Re} z > c\}$ for $c > 1$. Hint: Use the Weierstrass M-test.

(b) Show that

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \left(n^{-z} - \int_n^{n+1} x^{-z} dx \right)$$

and show the sum converges uniformly and absolutely on compact subsets of $\{z : \operatorname{Re} z > 0\}$. Hint: use the Fundamental Theorem of Calculus.

Probably the most famous problem in all of mathematics is to prove that if $0 < \operatorname{Re} z < 1$ and $\zeta(z) = 0$, then $\operatorname{Re} z = 1/2$.

9*. (a) Suppose p is a polynomial with all its zeros in the upper half plane $\mathbb{H} = \{z : \operatorname{Im} z > 0\}$. Prove that all of the zeros of p' are contained in $\mathbb{H}$. Hint: Look at the partial fraction expansion of p'/p.

(b) Use (a) to prove that if p is a polynomial then the zeros of p' are contained in the (closed) convex hull of the zeros of p. (The closed convex hull is the intersection of all half planes containing the zeros.)

10*. The goal is to show that the stereographic projection is conformal, i.e., preserves angles between curves.

First, fix a point $z^* \in S^2 \setminus \{N\}$ and consider the tangent plane $\mathcal{P}$ to S^2 at z^*. Let L be the line of intersection between $\mathcal{P}$ and $\mathbb{C}$. Also let ℓ be the line through N and z^* that meets the plane $\mathbb{C}$ in z. Show that ℓ makes the same angle with $\mathcal{P}$ as with $\mathbb{C}$.

Next, if λ^* is a line on the tangent plane $\mathcal{P}$ through the point z^*, draw the point T of intersection with the complex plane $\mathbb{C}$ (if it exists). Now draw the line λ in $\mathbb{C}$ from T to z. Also let P be the point of intersection between the line L and the vertical plane through N, 0, and z. Show that the triangle (z, T, P) is congruent to the triangle (z^*, T, P).

Finally, assume that λ_1^* and λ_2^* are two lines in $\mathcal{P}$ through z^*, meeting at an angle α. Assume without loss of generality that they meet $\mathbb{C}$ in T_1 and T_2 respectively. Construct λ_1 and λ_2 as above and show that they meet in z with the same angle α.

E-mail address: `pietro@math.ksu.edu`